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MATHEMATICAL BIOPHYSICS OF GROWTH, *by* N. RASHEVSKY.

MATHEMATICAL BIOPHYSICS OF CYTOPLASMIC STREAMING, *by* GALE YOUNG.

BULLETIN OF
Mathematical Biophysics

CONTENTS

ERRATUM

In the paper "The Mechanism of Cell Division" by N. Rashevsky, *Bull. Math. Biophysics.*, **1**, 23, 1939, the following somewhat confusing notation should be changed: substitute I for T in last term of equation (1); in left hand side of equation (3); in first line from top on page 25; in 12th and 19th line from top on page 27; in left hand side of equation (20).

In formula on 10th line from bottom on page 24 the expression in braces should be multiplied by c.

ELONGATION AND CONSTRICTION IN CELL DIVISION

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The deformation of a constant reaction ellipsoidal cell by diffusion and constant surface tension forces is studied. The critical size of a spherical cell at which it becomes unstable to ellipsoidal deformations is found to be the same as that obtained previously by N. Rashevsky from energy considerations. It is shown that such a cell once unstable will elongate to a finite amount, and that it will tend to constrict in the center and round up at the poles.

This paper makes an attempt to follow the deformation of a viscous body under the action of diffusion and surface tension forces. The treatment is in places admittedly rough and incomplete, but it is hoped that the work may present sufficient analogy with cell division phenomena to warrant its presentation. It is hoped also that it may serve to indicate the type of observations needed in connection with such theoretical studies. The dynamics of cell motion, which includes such phenomena as cell division, pseudopodic extensions and contractions, protoplasmic streaming, etc., involves essentially the deformation of a material system under the action of various forces. The system may offer both viscous and elastic resistance to the motion, while the driving forces may arise from a number of sources. Various writers have assigned the driving force to various causes, usually however without trying to calculate the rate and nature of the motion which would ensue. It is clear that this is an essential part of any theory, and it is important to attempt to do this, even though the problem is mathematically and physically rather difficult.

Among the more prominent views about the nature of the forces responsible for cell division is that of Butschli who regarded them as arising from a variation in surface tension over the cell surface, that of R. S. Lillie who considered them to be electrical in origin, and that of Heidenhahn who supposed that they arise from the pull of contractile fibers (Wilson, 1934, pp. 189-198). It has since become clear from the work of N. Rashevsky (1938a) that the drag forces exerted by diffusing substances play an important role in the behaviour of metabolizing systems.

The present paper will consider the action of diffusion forces and constant surface tension. The first gives rise to volume forces throughout the system of amount

$$\mathbf{F} = - \alpha \nabla c \tag{1}$$

where c is the concentration of the diffusing substance and α is a constant of the order of magnitude of RT/M, M being the molecular weight of the diffusing substance. The second gives rise to a normal pressure over the cell surface of amount

$$P = \gamma K \tag{2}$$

where γ is the surface tension and K is the mean curvature of the surface at the point in question.

Instability. It has been shown by Rashevsky that this combination of forces leads to a reasonable estimation of the size of living cells at which division might be expected to take place. First, by expanding the cell to infinite size and then recondensing it into two spheres each of half the original volume, it is possible to find a critical radius for the original cell above which the sequence of changes might occur spontaneously, at least insofar as concerns the overall work done in the process. It is supposed that α varies inversely as the cell volume during the operation, and the same for the rate, q, at which the diffusing substance is produced per unit volume. For a very large membrane permeability the resulting value for the critical radius (Rashevsky, 1938a, Chapter 8) is

$$\sqrt[3]{\frac{11.25 \, \gamma \, \overline{D}}{\alpha \, q}} \tag{3}$$

where $\overline{D}$ is some value intermediate between the values of the internal and external diffusion coefficients D_i and D_e.

Secondly, (Chapter 9) Rashevsky calculates the work done by these forces in infinitesimal deformations of the cell from the spherical shape. In particular we may consider, for reasons to appear presently, such a deformation which makes the cell into an ellipsoid of revolution. The equation in polar coordinates of an ellipsoid with semi-major axis a, semi-minor axis b, and focal separation $2f$, is

$$r = \frac{b}{\sqrt{1 - e^2 \cos^2 \theta}} = b \left(1 + \frac{e^2 \cos^2 \theta}{2} + \cdots \right), \tag{4}$$

where $e = f/a$ is the eccentricity. Making use of the relation

$$\cos^2 \theta = \frac{2}{3} P_2 (\cos \theta) + \frac{1}{3} P_0 (\cos \theta),$$

we find in Rashevsky's notation that for e very small

$$r_0 = b$$

$$\Delta_0\, a_2 = \frac{1}{3}\, b\, e^2 ,$$

(5)

while the other coefficients a_n are negligible. Substituting these values in his general expression (eq. 94, p. 97) and setting $h = \infty$, we find that the cell is unstable to such a deformation if its radius is larger than

$$\sqrt[3]{\frac{6\,\gamma\,(2\,D_i + 3\,D_e)}{q\,\alpha}} .$$

(6)

The difference between the two derivations should be carefully noted. Expression (3) gives an estimate for the radius at which the energy change from initial to final state becomes favorable to division; while (6) gives the radius at which the work balance is favorable in a small deformation of the cell into an ellipsoid of revolution. They do not measure the same thing, but it may be noted that they agree well. If $\bar{D} = D_e >> D_i$, then (6) gives a value 1.17 times that of (3); if $\bar{D} = D_e = D_i$, then (6) is 1.39 times as large.

Neither of the above developments concerns itself with intermediate stages of division, or gives any assurance that division will proceed. To consider these matters it is necessary to discuss in more detail the mechanics of cell deformation. The first stage of division in many cells is typically an elongation into an ellipsoid of revolution, the so-called "karyokinetic" elongation (Wilson, 1934, p. 157). This reaches a maximum value, and the cell is then divided by a furrow constricting inwards in its equatorial plane. Similarly the early stages of nuclear division are often marked by an approximately ellipsoidal shaped spindle, which later takes on a dumb-bell shape as the central region constricts.

For consideration of the early stages of division it is thus appropriate to study ellipsoidal shaped cells. This is why an ellipse was used in deriving (6). Considering the cell (or spindle) as an incompressible viscous body of viscosity η, and neglecting the resistance of the bounding membrane to the passage of the diffusing substance, it is found (Young, 1939) that the contribution to the average relative elongation along the major axis due to the internal diffusion drag is

$$\bar{e}_D = \frac{2}{45}\, \frac{\alpha\, q}{\eta}\, \frac{a^2 - b^2}{2\,D_i + m\,D_e} ,$$

(7)

where m is a function of a/b as shown on the accompanying graph. (Fig. 1). Similarly the contribution of the surface tension is

$$\bar{e}_{s.r.} = -\frac{\gamma}{3\,\eta}\,\beta\,\frac{a-b}{a\,b}\,, \tag{8}$$

where β is a function of a/b :

$$\beta(1) = 1.6 \; ; \quad \beta(2) = 1.5 \; ; \quad \beta(\infty) = 1.18\,.$$

Interpreting $\bar{e} = \bar{e}_D + \bar{e}_{s.r.}$ *just* > 0 to mean the beginning of elongation, the critical radius is again found to be that given by (6). Thus there is complete agreement with Rashevsky's energy treatment as regards the size of cell at which instability to infinitesimal ellipsoidal deformations sets in. With the present approach, however, it is possible to go farther and follow up to finite elongations — as far, in fact, as the cell approximates to ellipsoidal shape.

Finite elongation. It will first be shown that if the radius exceeds the critical value given by (6), then the cell will undergo a finite elongation. In a constant volume ellipsoidal deformation

$$ab^2 = \text{constant} = k = \frac{3\,V}{4\,\pi}\,, \tag{9}$$

so that (8) may be written as

$$\bar{e}_{s.r.} = -\frac{4\,\pi\,\gamma}{9\,V\,\eta}\,\beta\,b\,(a-b)\,. \tag{10}$$

Also (7) may be expressed as

$$\bar{e}_D = \frac{2}{45}\,\frac{\alpha\,q}{\eta}\,\frac{1+a/b}{2\,D_i + m\,D_e}\,b\,(a-b)\,. \tag{11}$$

Denoting

$$2\,D_i = x\,D_e\,, \tag{12}$$

the ratio of (11) to the absolute value of (10) is

$$r = y\,\frac{1+a/b}{\beta}\,\frac{1}{x+m}\,, \tag{13}$$

where

$$y = \frac{\alpha\,q\,V}{10\,\pi\,\gamma\,D_e}\,. \tag{14}$$

The first variable term in (13) has the value 5/4 at $a = b$ and the value 2 at $a = 2b$. Instability occurs at $r = 1$; i.e., at

$$y = \frac{4}{5}(x+3). \tag{15}$$

To see if elongation will continue for this value of y it is necessary to compare

$$\frac{4}{5}(x+3)\frac{1+a/b}{\beta} \tag{16}$$

and

$$x+m. \tag{17}$$

As long as (16) exceeds (17) the elongation will continue. The former has the value $x+3$ at $a=b$, the value $\frac{8}{5}(x+3)$ at $a=2b$. It is thus approximately a straight line of slope $\frac{3}{5}(x+3)$; which for comparison with (17) is to be drawn on the m-graph with the value 3 at $a=b$.

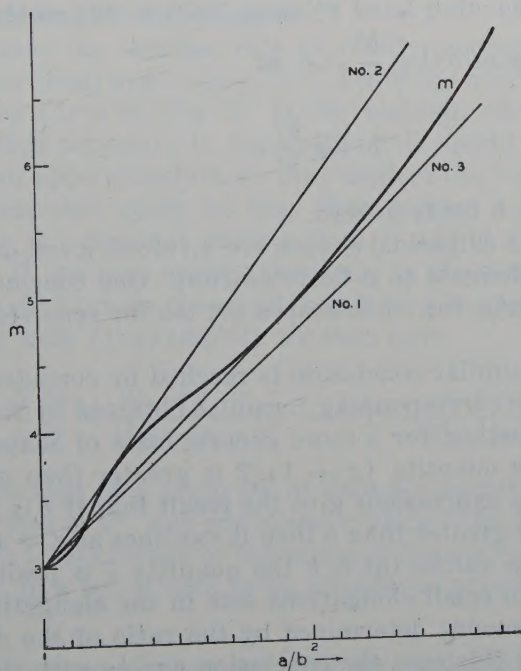The minimum slope of the (16) line occurs for $x=0$ (i.e.

FIGURE 1

$D_e >> D_i$) and is $\frac{9}{5}$. This is No. 1 on the graph and is seen to be above (17) up to $a = 1.14\,b$, so that elongation once begun will continue to this extent.

As x increases the straight line swings upward, increasing the ratio of a/b at which elongation will cease. Up to position No. 2 this increase is gradual up to $a = 1.3b$. Then, however, the equilibrium ratio suddenly jumps to about $a = 5.76b$ as the hump in the m-curve is surmounted. It always remains finite, however, as will be seen later. This jump occurs at a slope of 2.76, which requires $x = 1.6$ or $D_i = 0.8\,D_e$.

A line of slope between that of No. 3, which corresponds to $x = 0.37$ or $D_i = 0.185\,D_e$, and that of No. 2 will intersect the m-curve in 3 points. The middle of these is unstable, the top one stable (supposing, of course, that the formulae derived for an ellipsoid remain valid).

To see that the elongation (with the same supposition) would always remain finite, note that for $a >> b$ the surface tension restoring effect (10) varies as ab, i.e. as $\sqrt{ka}$ which becomes infinite as a increases. On the other hand (Young, 1939, p. 49) as the elongation continues $\bar{e}_D$ varies as $b^2 \log \frac{b^2}{a^2}$, i.e. as

$$b^2 \log \frac{b^6}{k^2}, \qquad (18)$$

which vanishes as b tends to zero.

Thus as far as ellipsoidal shapes are involved, a cell once unstable will continue to elongate to a finite amount. One concludes that the same is probably true for other shapes not too far removed from ellipsoids.

A somewhat similar conclusion is reached by considering instead of (7) and (8) the corresponding formulae obtained by Rashevsky by an approximate method for a more general class of shapes (Rashevsky, 1939). If the quantity $(x + 1)/2$ is greater than unity, where x is as in (12), his expressions give the result that if $\bar{e}$ is just greater than zero at a just greater than b then it vanishes at $a = (x + 1)b/2$. Between these two values for a/b the quantity $\bar{e}$ is positive, so that when instability to small elongations sets in the elongation will continue to a finite amount determined by the ratio of the diffusion coefficients. Thus in this case the conclusion agrees with that obtained above from the study of ellipsoidal cells. For $x = 1.6$ both treatments give $a = 1.3\,b$ for the amount of elongation.

On the other hand if $x < 1$, i.e. if $D_i < \frac{1}{2} D_e$, then the other equilibrium ratio in Rashevsky's treatment, namely $a = (x+1)b/2$, corresponds to a flattened or shortened shape body. When the spherical shape is just unstable to small shortenings the shortening will continue to a finite amount. The above treatment of ellipsoids has been confined to prolate or elongated ellipsoids, and hence can not be used to discuss the case of $b > a$. Extension to oblate or shortened ellipsoids would permit comparison with Rashevsky's results in this case also. This will not be entered into here.

Rate of elongation. If $D_i < 0.8 D_e$, which will usually be the case for entire cells though perhaps not for mitotic spindles, then a reaction rate just large enough to make the spherical shape unstable will not carry the ellipsoidal elongation beyond $a = 1.3\,b$ at the most. In some instances, particularly in division of entire cells, the early elongation does remain within some such small value. In other cases however, and usually so with spindles, the elongation exceeds this value while still remaining approximately ellipsoidal. If $D_i < 0.8 D_e$ and the elongation does progress beyond $a = 1.3\,b$, then the reaction rate, if supposed constant throughout the cell, must have increased since the beginning of the elongation.*

In any case the average rate of relative elongation at any moment is proportional to $\bar{e} = \bar{e}_D + \bar{e}_{s.T.}$, i.e. to the height of the straight line above the curve in Fig. 1. If the reaction rate is considerably larger than that necessary to instability, this height will be considerable and as an approximation we may neglect the bends in the curve. Over a considerable range we may then replace m by $3a/b$ in (7). This quantity is somewhat greater than 2, and if D_e is considerably greater than D_i we may neglect the latter in the denominator of (7). Similarly we shall ignore the small change of β in (8) and set $\beta = 1.5$. Identifying $\bar{e}$ with $(1/a)\,(da/dt)$ we then have

$$\frac{1}{a}\frac{da}{dt} = \frac{2}{135}\frac{\alpha q}{\eta D_e}\frac{b}{a}\,(a^2 - b^2) - \frac{\gamma}{2\,\eta}\frac{a-b}{ab}. \tag{19}$$

It may be that in a given set of data the variation in b^2 is small compared to that of a^2, and one might replace b^2 by ka^2 in the $a^2 - b^2$ term, where k is an average of the ratios at the ends of the data range. For example, in a set of data to be considered presently b^2 varies be-

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* In the present paper we are supposing the reaction rate constant throughout the cell, but it is not insisted that this is the case. There are in fact reasons for thinking that astral systems represent mechanically important localized reaction centers, and one would surely want to study them as such. Here, however, we are tracing consequences of the constant reaction assumption.

tween $.4\,a^2$ and $.3\,a^2$, and we shall take $k = 0.35$. With this simplification (19) reduces to

$$\frac{da}{dt} = \frac{2}{135}\,\frac{\alpha\,q\,(1-k)}{\eta\,D_e}\,b\,a^2 - \frac{\gamma}{2\,\eta}\left(\frac{a}{b}-1\right). \tag{20}$$

In a constant volume ellipsoidal deformation

$$\frac{4}{3}\pi\,a\,b^2 = V = \text{constant}, \tag{21}$$

and (20) becomes

$$\frac{da}{dt} = A - B\,a^{3/2}. \tag{22}$$

where

$$A = \frac{\gamma}{2\,\eta} \tag{23}$$

and

$$B = \frac{\gamma}{2\,\eta}\sqrt{\frac{4\,\pi}{3\,V}} - \frac{2}{135}\,\frac{\alpha\,q\,(1-k)}{\eta\,D_e}\sqrt{\frac{3\,V}{4\,\pi}}. \tag{24}$$

We shall attempt to apply these results to some observations of Bělař (1929) on the elongation of mitotic spindles. There are a number of objections to this application, of which but two will be mentioned. The first is that the spindle volume is not constant in Bělař's material, so that the increase in length comes about from growth as well as stretching, which would tend to exaggerate the velocities. The second is that the spindle is not a homogenous viscous body. Following Rashevsky (1938b) we shall attempt to reduce this difficulty by considering only his data after the chromosomes have separated, which should decrease to some extent the effect of possible contractile or elastic fiber pull.

Bělař's data gives spindle length as a function of time. It was first smoothed by fitting a curve to it (Rashevsky, 1938b, Fig. 5), and then the values of a and da/dt were taken from the curve. For any two points (22) gives by subtraction

$$\left(\frac{da}{dt}\right)_1 - \left(\frac{da}{dt}\right)_2 = -B\,(a_1{}^{3/2} - a_2{}^{3/2}), \tag{25}$$

so that each pair of points gives a value for B. For simplicity each other point was paired with the second point, and the resulting values averaged to give B. Using this average B in (22) gives an A value for each point, and these were similarly averaged. The results are

given in the accompanying table, where the last column is obtained by substituting the average B and A values back into (22). These values are

$$A = 1.77 \times 10^{-6}$$

$$B = 0.036 \, .$$

(26)

observed $2a$ in 10^{-4} cm.	smoothed $2a$ in 10^{-4} cm.	smoothed da/dt in 10^{-6} cm/sec	calculated da/dt in 10^{-6} cm/sec
21.4	21.8	0.51	0.48
24.4	24.3	0.22	0.24
25.2	25.4	0.11	0.15
25.6	25.9	0.083	0.10
26.6	26.4	0.07	0.04
27.0	27.0	0.008	—0.03

The volume V is obtained by averaging the initial and final values, much as was done above for k ;

$$V = 2.6 \times 10^{-9}$$

$$k = 0.35 \, .$$

(27)

Using these values in (23) and (24) we obtain

$$\frac{\gamma}{\eta} = 3.54 \times 10^{-6}$$

(28)

and

$$\frac{\alpha \, q}{\eta \, D_e} = 1.4 \times 10^{5} \, .$$

(29)

Taking

$$\alpha = \frac{RT}{M}$$

$$M = 100$$

(30)

$$D_e = 10^{-6}$$

gives

$$\frac{q}{\eta} = 5.7 \times 10^{-10}$$

(31)

Using (28) and (29) in (6) with $D_e >> D_i$ gives for the critical radius 7.7×10^{-4} cm., while the volume of (27) corresponds to a radius of 8.57×10^{-4} cm. for the spindle in spherical shape. As mentioned above this would, in the present treatment, indicate that

the reaction rate is greater than required to cause instability of the spindle in spherical shape.*

Constriction. The above considerations hold only insofar as the cell remains approximately ellipsoidal in shape, and their application depends upon the empirical fact that cells and spindles do have ellipsoidal stages. So far the theory has accepted the shape as given, without trying to see what sequence of shapes would actually be expected. By the use of Betti's formula (Young, 1939) the average rate of elongation of an incompressible viscous body under the action of any assigned forces can be determined, as is illustrated by expressions (17) and (8). This however gives little information as to the change in shape of the body; it might be elongating in some regions and constricting in others. For application such as to cell division it is necessary to study different regions of the body in greater detail.

To this end consider a slab of the body cut out by two parallel planes. If we can estimate the average rate of elongation in such an arbitrary slab we shall have more information about what is happening in different parts of the body. Again considering only the effect of internal diffusion drag and constant surface tension, the forces acting on the shaded portion of the body (Fig. 2) are (a) surface tension pressure over its lateral surface S_3 ; (b) its internal diffusion drag forces; and (c) the stresses exerted across S_1 and S_2 by the end regions of the body.
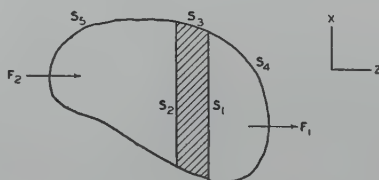

FIGURE 2

In determining by Betti's formula the average elongation $\bar{e}$ of the slab in the z direction the effect of the drag forces (b) is the same (Young, 1939) as if a pressure αc were acting normally over $S_1 + S_2 + S_3$. Assuming for simplicity that the system is symmetric about the z axis, the (c) forces involve the integral

$$\iint_{S_1+S_2} (z\, Z' - x\, X')\, ds , \tag{32}$$

where Z' and X' are the stress components exerted by the end regions

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* This of course does not imply that the spindle ever did have a spherical shape, it may have come into being in an elongated form.

across unit area. The Z' part is readily evaluated since z is constant on S_1 and on S_2. Thus

$$\iint_{S_1} z\, Z'\, ds = z_1 \iint Z'\, ds = z_1 F_1 \tag{33}$$

where F_1 is the total z force acting on the right hand end region; namely $F_1 =$ (total diffusion force on the end) — (lateral surface tension holding the end on). The first term of F_1 is as if a pressure αc were acting over $S_2 + S_4$. The S_1 part of this cancels the S_1 part of the (b) term, and similarly for S_2. Thus we are left with the following contributions:

(I) pressure αc acting on S_3

(II) z_1 [(z force of pressure αc on S_4) +
        (z surface tension pull on end no. 1)]

(III) z_2 [(z force of pressure αc on S_5) +
        (z surface tension pull on end no. 2)]

(IV) $- \iint_{S_1+S_2} x\, X'\, ds$

(V) surface tension pressure acting over S_3.

Note that a constant added to c will cancel out from (I), (II), and (III), because in (II) a constant normal pressure over S_4 gives the same total z force on end no. 1 as if it were acting normally (from the right) over S_1, and similarly for S_5 and S_2. Then with (I) the effect on the slab is of a constant normal pressure over its entire surface, which contributes nothing to $\bar{e}$. Thus only variable terms need be retained in c.

Constriction in ellipsoid. We shall first apply these considerations to an ellipsoid of revolution to see what is the tendency for departure from that shape. In this we shall make use of notation, methods, and results previously given (Young, 1939). The longer axis is along z ; the semi-major and semi-minor axes are denoted respectively by a and b ; the distance of the foci from the center is $f = \sqrt{a^2 - b^2}$; and z distance from the center is $z = a \cos \theta$. On the ellipsoidal surface the concentration is of the form

$$c = G_0 + G \cos^2 \theta , \tag{34}$$

wherein, as mentioned above, the G_0 term may be ignored. The contribution of term (I) to $\bar{e}$ is

$$-\frac{\alpha}{3\eta V} J ; \tag{35}$$

where

$$J = \iint c [z (\cos n\, z) - x \cos (nx)]\, ds , \tag{36}$$

and where V is the volume of the slab under consideration. Carrying out the integration gives

$$J = \pi\, G\, a\, b^2 \{\frac{1}{3}[\cos^3\theta]_1^2 - \frac{3}{5}[\cos^5\theta]_1^2\}$$

$$= \frac{\pi}{3}\, G\, \frac{b^2}{a^2}\, (z^3_2 - z^3_1) - \frac{3\,\pi}{5}\, G\, \frac{b^2}{a^4}\, (z^5_2 - z^5_1),$$

$$(37)$$

where the subscripts 1 and 2 refer to the two ends of the slab.

The total z force of pressure $\alpha\, c$ acting over S_4 is given by

$$- \alpha \int\int_{S_4} c \cos(nz)\, ds\ ;$$

$$(38)$$

which, apart from the G_0 term, turns out to be

$$\frac{\pi}{2}\alpha\, G\, b^2[\cos^4\theta]_0^1 = \frac{\pi}{2}\,\alpha\, G\, \frac{b^2}{a^4}(z^4_1 - a^4).$$

$$(39)$$

The z component of surface tension holding end no. 1 onto the slab is

$$- 2\,\pi\,\gamma\, \frac{ab\,\sin^2\theta_1}{\sqrt{a^2 - f^2\cos^2\theta_1}}.$$

$$(40)$$

Similarly the total z force acting on end no. 2 is the sum of

$$\frac{\pi}{2}\,\alpha\, G\, \frac{b^2}{a^4}(a^4 - z^4_2)$$

$$(41)$$

and

$$2\,\pi\,\gamma\, \frac{ab\,\sin^2\theta_2}{\sqrt{a^2 - f^2\cos^2\theta_2}}.$$

$$(42)$$

Finally the contribution to $\bar e$ of surface tension over S_3 is found to be

$$\bar e = -\frac{\gamma}{3\eta v}\pi\, b\,[- (\frac{3}{2}\frac{a^4}{f^2} - 2\frac{a^2}{f})\times$$

$$(\sin^{-1}\frac{f\, z_1}{a^2} - \sin^{-1}\frac{f\, z_2}{a^2})$$

$$+ (3\frac{a^2\, b^2}{f^2} - b^2)\times$$

$$(\frac{z_1}{\sqrt{a^4 - f^2\, z^2_1}} - \frac{z_2}{\sqrt{a^4 - f^2\, z^2_2}})$$

$$(43)$$

$$-\frac{3}{2\,f^2}(z_1\sqrt{a^4-f^2\,z^2{}_1}-z_2\sqrt{a^4-f^2\,z^2{}_2})\,].$$

These expressions simplify considerably if the slab is made very thin, so that $z_1 - z_2 = dz$. Thus (37) becomes, by differentiation,

$$J = \pi\,G\,b^2\,(3\,\frac{z^4}{a^4}-\frac{z^2}{a^2})\,dz \; ; \tag{44}$$

which with (35) and

$$V = \pi\,b^2\,\frac{a^2-z^2}{a^2}\,dz \tag{45}$$

gives a contribution

$$\bar{e} = \frac{\alpha\,G}{3\,\eta}\frac{z^2}{a^2}\frac{a^2-3\,z^2}{a^2-z^2}. \tag{46}$$

The terms (39) and (41) contribute

$$\bar{e} = \frac{1}{3\,\eta\,V}[z_1\,(39) + z_2\,(41)]. \tag{47}$$

The value of the bracket comes out to be

$$\frac{\pi}{2}\,\alpha\,G\,\frac{b^2}{a^4}\,[z^5{}_1 - z_1\,a^4 + z_2\,a^4 - z^5{}_2], \tag{48}$$

which for a thin slab reduces to

$$\frac{\pi}{2}\,\alpha\,G\,\frac{b^2}{a^4}\,(5\,z^4 - a^4)\,dz\,. \tag{49}$$

In this last case (47) becomes

$$\bar{e} = \frac{\alpha\,G}{3\,\eta}\frac{1}{2\,a^2}\frac{5\,z^4 - a^4}{a^2-z^2}. \tag{50}$$

The term (IV) can not be directly evaluated from a knowledge of the forces applied to the entire body, as the X' relate to the state of internal stress set up. For the present this term will be ignored. With this neglected the total $\bar{e}$ in the slab due to diffusion forces is the sum of (46) and 50), namely

$$\bar{e}_D(z) = -\frac{\alpha\,G}{3\,\eta}\frac{a^2-z^2}{2\,a^2}. \tag{51}$$

Similarly (43) reduces, by differentiation, for a narrow slab to

$$\bar{e} = -\frac{\gamma}{3\,\eta\,V}\,\pi\,b\,dz\,\frac{1}{(a^4 - f^2\,z^2)^{3/2}}\,[-a^6 - a^4\,b^2 +$$

$$(6\,a^4 - 2\,a^2\,f^2)\,z^2 - 3\,f^2\,z^4]\ ;$$ (52)

while the term

$$\bar{e} = \frac{1}{3\,\eta\,V}\,[z_1\,(40) + z_2\,(42)]$$ (53)

becomes

$$\bar{e} = -\frac{2\,\pi\,\gamma}{3\,\eta\,V}\,b\,dz\,(a^4 - f^2\,z^2)^{-3/2}\,(a^6 - 3\,a^4\,z^2 + 2\,f^2\,z^4).$$ (54)

Adding (47) and (49) and introducing (45) gives for the total surface tension term, (IV) being neglected as before,

$$\bar{e}_{S.T.}\,(z) = -\frac{\gamma}{3\,\eta}\,\frac{a^2\,f^2}{b}\,\frac{a^2 - z^2}{(a^4 - f^2\,z^2)^{3/2}}.$$ (55)

It is seen that both (51) and (55) are numerically largest at the equator ($z = 0$) and drop off to zero at the poles ($z = \pm\,a$). They decrease, however, at different rates, and this gives rise to an interesting consequence. With G negative expression (51) is positive, showing a tendency for each part of the body to elongate; while (55) is always negative, showing that surface tension tends to thicken each part of the body. But (51) drops off toward the poles more rapidly than does (55), so that the thickening or rounding up tendency is relatively greater at the poles, while the elongating tendency is relatively greater at the equator. Since the neglected term (IV) involves transverse stresses which are opposite on two adjacent slabs, its retention would tend to smooth out the variation in $\bar{e}\,(z)$ from poles to equator.

In particular if $\bar{e}$ for the entire body is zero, then there will be constriction at the center and thickening at the ends. Expression (7) is (Young, 1939)

$$\bar{e}_D = -\frac{2}{15}\,\frac{\alpha\,G}{\eta},$$ (56)

and thus (7) plus (8) equals zero if

$$G = -\frac{5\,\gamma\,\beta}{2\,\alpha}\,\frac{a - b}{ab}.$$ (57)

With this value of G the sum of (51) and (55) is

$$\bar{e}\,(z) = -\frac{(a^2 - z^2)\,(a - b)\,\gamma}{3\,\eta\,b}\left[\frac{a^2\,(a + b)}{(a^4 - f^2\,z^2)^{3/2}} - \frac{5\,\beta}{4\,a^3}\right],$$ (58)

which changes sign at $\cos \theta = z/a$ satisfying

$$e^2 \cos^2 \theta = 1 - \left[\frac{4}{5\beta} \left(1 + \frac{b}{a}\right) \right]^{2/3} , \qquad (59)$$

where $e = f/a$. The central region of the cell up to values of z given by (59) will be constricting and elongating; the end regions for greater values of z will be broadening and shortening.

For example consider the case $a = 2b$ and thus $e^2 = \frac{3}{4}$, $\beta = 1.5$. Then (59) gives $z = \pm 0.43\, a$, so that the ellipsoid is constricting over a little less than half its length, and rounding up over the remaining. If $a = 1.34\, b$, with $e^2 = 4/9$ and $\beta = 1.57$, then $z = \pm 0.31\, a$.

In the above, as in (19), $\dfrac{1}{a} \dfrac{da}{dt}$ was identified with the volume average $\bar{e}$ over the entire cell interior. Instead of employing (7) or (56) as the diffusion contribution to $\dfrac{1}{a} \dfrac{da}{dt}$, one might use

$$\bar{\varepsilon}_D = \frac{1}{a} \int_0^a \bar{e}_D(z)\; dz , \qquad (60)$$

with $\bar{e}(z)$ as given by (51). This comes out to be

$$\bar{e}_D = -\frac{1}{9} \frac{a\, G}{\eta} , \qquad (61)$$

which differs slightly from (7) in the numerical factor. In the same way one might consider, instead of (8),

$$\bar{\varepsilon}_{S.T.} = \frac{1}{a} \int_0^a \bar{e}_{S.T.}(z)\, dz = \frac{\gamma}{3\,\eta\, a} \left(1 - \frac{a^2}{bf} \sin^{-1} \frac{f}{a}\right) . \qquad (62)$$

This may be written as

$$\bar{\varepsilon}_{S.T.} = -\frac{\gamma}{3\,\eta} \xi \frac{a-b}{a\,b} , \qquad (63)$$

where ξ is a function of a/b; $\xi\,(1) = 1.33$, $\xi\,(2) = 1.4$, $\xi\,(\infty) = 1.57$. Using (62) plus (61) gives the same critical radius (6) as using (7) plus (8), and in general the results are closely comparable.

Constriction in other shape bodies. It has been seen above that as the body elongates into ellipsoidal shape it tends to pinch in at the equator and bulge at the poles, which is reminiscent of the actual course of cell and spindle division. As soon as the constriction has proceeded to any appreciable degree the above formulae cease to hold,

and must be revised. It does not appear feasible to attempt to follow the complete course of the deformation, but we may make some rough estimates. Suppose the constriction to have proceeded to the stage of the diagram (Fig. 3), where the body is comprised of two spheres
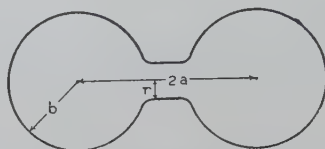

FIGURE 3

connected by a narrow neck. The diffusion problem is now somewhat like that of two separate spheres, and thus (Young, 1936) the force tending to separate the spherical ends is roughly given by

$$F = \frac{2\,\pi\;a\,q}{3\;D_e}\frac{b^6}{4\,a^2}.\tag{64}$$

The lateral surface tension force tending to hold each end onto the neck is of the order of

$$2\,\pi\,r\,\gamma\;;\tag{65}$$

while over the lateral surface of the neck itself is a normal surface tension pressure of about

$$\frac{\gamma}{r}\tag{66}$$

As the spheres tend to pull apart the narrowest region of the neck constricts inward, much like the "necking" of an iron rod in a testing machine. Introducing the above forces into Betti's equation for the central part of the neck alone, and ignoring its internal diffusion forces, we obtain

$$\bar{e} = \frac{1}{3\,\pi\,\eta}\frac{F - \pi\,r\,\gamma}{r^2}.\tag{67}$$

Now

$$\bar{e} = -\frac{2}{r}\frac{dr}{dt},\tag{68}$$

so that if a and b be regarded as remaining roughly constant while the constriction proceeds we have

$$\frac{dr}{dt} = M - \frac{N}{r},\tag{69}$$

where

$$M = \gamma/6\,\eta$$

$$N = \frac{\alpha\,q\,b^6}{36\,\eta\,D_e\,a^2}.$$

(70)

At this rate r would drop to zero in a finite time, provided that it has started to decrease at all. If $N/r >> M$, the decrease is according to

$$\frac{dr}{dt} = -\frac{N}{r}\,;$$

(71)

so that from a finite initial value r_0 the radius will have dropped to zero in a time.

$$t = \frac{r_c^2}{2\,N}.$$

(72)

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ON THE SO CALLED PLASTICITY OF THE CENTRAL NERVOUS SYSTEM

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The taking over of a function of a region of the brain by another region, in case of brain injuries, is discussed in its mathematico-biophysical aspects.

One of the most important properties of the central nervous system is a peculiar combination of definitely localized structures on one hand and of a large amount of variability of function of a given structure on the other. Removal of a region of the brain, which is known to control a given activity, usually results in the loss of that activity. But not infrequently the loss is temporary, the lost function being eventually assumed by some other part of the brain, which normally may not appear to contribute anything to that particular function. It is the purpose of this note to point out, that such phenomena find a very natural and simple explanation in the light of the mathematico-biophysical theory of the brain, developed by the author in his book (Rashevsky, 1938a) and by the author and others in subsequent publications (Landahl, 1938, Rashevsky, 1938b, Householder, 1939).

In these publications the following general structure has been considered and applied to various special cases. A number of parallel excitatory neuronic chains are inter-connected with each other by inhibitory fibers, so that the excitation of one neuronic chain tends to inhibit the others. As has been shown (Rashevsky, 1938 a, b, Householder, 1939) such an arrangement possesses the following property: If all the parallel chains are excited with unequal intensity at their afferent ends, then the excitation is transmitted only through a few of these chains, which are excited most strongly, all the weaker ones being completely inhibited, even though they are stimulated with a super-liminal inbensity. How many of the chains will be inhibited depends on the relative values of excitation of the different chains, and on the constants characteristic of the inhibitory fibers.

Suppose now that a given end organ is innervated by a number of central neurons, spontaneously excited (Rashevsky, 1938a, Chapt. XXIV), which are scattered throughout the whole cortex, or through a

very large region K_0 of the latter. Let however these neurons be not identical in their physical properties, but possess different intensities of excitation. Let the most strongly excited neurons lie more densely around a point $P(x_0, y_0, z_0)$, which is contained in K_0. The density of the more strongly excited neurons may fall off continuously from a certain point P to the other parts of K_0. We may say then that to each point x, y, z in K_0 corresponds a given average intensity of total excitation $E = f(x,y,z)$, where $f(x,y,z)$ has a maximum for $x = x_0$, $y = y_0$, $z = z_0$. If each of these neurons sends off inhibitory fibers to all others, then under these conditions (Rashevsky, 1938b) all neurones, having an intensity of excitation E less than a certain value E_1, will be inhibited, while those with $E > E_1$ will remain excited. But those latters lie in a *sharply limited* region K_1 around P, the boundary of this region being given by $f(x,y,z) = E_1$. Application of a stimulus to region K_1, which will change the excitatory state of the neurones contained therein, will result in a change of the state of the end organ. The application of a similar stimulus to any point outside of K_1, but within K_0, will not result in any change of state of the end organ, because of the complete inhibition of the neurones involved, by the more strongly excited neurones of K_1. Thus, such a "functional" test will reveal a sharp localization of function. Let however the region K_1 be removed. The inhibitory action of the neurones contained in K_1 upon the remaining ones is thus also removed and some of these remaining neurones will become now excited and appear functional. Again however not all of the remaining neurones will carry on the excitation, but only a small group of the most strongly excited ones. This group is again localized in a sharply defined region K_1'. If the function $f(x,y,z)$ falls off monotonically with increasing distance from P, then the functions of a removed region will be taken over by an adjacent region. But if $f(x,y,z)$ has several relative maxima, K_1' may lie quite far apart from K_1. In general we should expect, that the extent of damage to K_1 would affect the size of K_1', depending on the shape of the function $f(x,y,z)$. It should be possible in principle to determine the shape of $f(x,y,z)$ by a quantitative study of the effects of brain injuries of different extent.

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CONTRIBUTIONS TO THE MATHEMATICAL BIOPHYSICS
OF THE CENTRAL NERVOUS SYSTEM

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Various neural mechanisms are considered which deal with point to point correspondence between two sets of neural elements with a smaller number of conducting elements between them; the transmission of nerve impulses in a limited range of intensities; movement of the transmission of excitation along a contour; the reaction to the size of an object independent of its distance; and an interpretation of the effect of a warning stimulus and of stimulus intensity upon reaction time. For the latter cases a comparison of the theoretical equations is made with some of the available experimental data, and a general agreement is found.

In discussing the following nerve mechanisms we shall use the concepts and notation developed by N. Rashevsky (1938, Ch. XXII), unless specific reference is made to the contrary. If the afferent processes of a nerve fiber are stimulated by an intensity, S, greater than a threshold, h, excitation occurs and the intensity of the excitation, E (proportional to the frequency and strength of the individual nerve impulses), increases in some manner with the strength S of the stimulus. At the efferent processes of the nerve fiber, an excitatory factor, ε, or inhibitory factor, j, (or both) is produced at a rate proportional to the excitation, E, and is dissipated proportional to its own concentration. The difference $\varepsilon - j$ acts as a stimulus on the next order nerve fiber if $\varepsilon - j$ is greater than the threshold, h, of the latter fiber.

I. Point to point correspondence between two sets of neural elements with a smaller number of conducting elements between them. It is well known that there are more photo-sensitive elements on the retina than there are transmitting elements in the optic nerve (Maximow and Bloom, 1938, Ch. XXVII). Only certain cones are connected through bipolar neurons to ganglion cells in a one-to-one relation (b—h—s), though the third order neurons are influenced by other cones (Maximow and Bloom, 1938, p. 601). If a group of rods or cones stimulates a single ganglion cell and the result is only an increased sensitivity, no difficulty would arise. However, some evidence

indicates that a higher degree of discrimination is possible between elements in the retina than would be indicated by the number of fibers in the optic nerve. The net result would be that a group of neurons is able to transmit excitation to corresponding members of another group through a single fiber or smaller number of fibers than in the initial or final group. Such a mechanism could conceivably occur elsewhere than in the optic system. For convenience we shall discuss the problem as though the phenomenon occurred in the optic system.
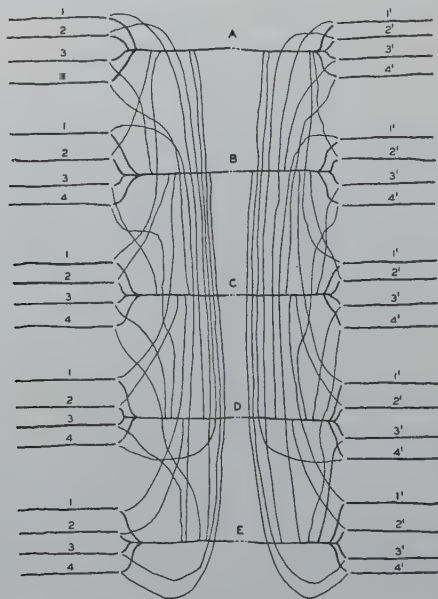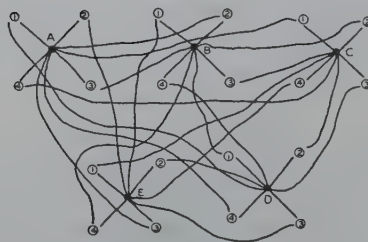

FIGURE 1

FIGURE 2

An interpretation of such a phenomenon must not involve the use of hypotheses which are contradicted by known facts regarding neurons and their behavior, or the use of specific assumptions, unless the latter are indicated directly by more fundamental processes. One might otherwise assume a specific type of action by each member of

the first group which could be transmitted only by the corresponding member of a higher group; the specific type of action might be a certain frequency, intensity or pattern of both. The well known work of P. Weiss (1936) seems to indicate specificities in groups of fibers. We shall, however, attempt a solution without the use of such an assumption.

Let us now discuss the mechanism in Figure 1 which exhibits the property of apparent point to point correspondence between two sets of nervous elements with a smaller number of intermediate connecting fibers, but which involves neurons having properties assumed from other lines of evidence. The mechanism was suggested while studying Dr. S. Polyak's semi-schematic diagram of the retina (Maximow and Bloom, 1938, Fig. 519), although there may seem to be little in common between Figure 1 and Dr. Polyak's diagram.

In order to have a figure which is not entirely too complex for illustration, we have in Figure 1, five groups of only four fibers ($A1$, $A2$, $\cdots$, $B1$, $\cdots$). Each of the four fibers in a group make synaptic contact with a common fiber (A, B, $C \cdots$) which in turn makes synaptic contact with each of four third order fibers ($A1'$, $A2'$, $\cdots$, $B1'$, $\cdots$). Let each first order fiber ($A3$) also make synaptic contact with an additional afferent fiber (B) which invariably is a collateral from a different group, and let this collateral afferent fiber have a symmetric collateral efferent fiber which connects with the third order fiber ($A3'$) corresponding to the first order fiber. Let us also assume that the threshold of the third order fibers is such that excitation does not occur unless both efferent fibers which synapse with it are excited together. For convenience we assume that all fibers, having the same function, are identical, though it is not necessary (Hecht, 1927-1928). Also the omission of the bipolar cells from consideration may have introduced an unnecessary restriction.

It is possible to satisfy the required conditions in at least two ways. In Figure 1 a closed system is shown, and it is also possible to construct similar systems which are complete to any multiple of the above. That is, with four fibers in a group, a complete system would involve forty first order fibers, ten second order and forty third order fibers, the ratio still remaining four to one. Let us consider only the simple case shown in the figure. The efferent end of every first order fiber, as $A3$, connects with a second order fiber of its own group, A, and an afferent collateral from another group, B; and no two fibers in the same first order group synapse with a collateral from the same second order group. The cyclic arrangement in the figure is quite evident upon inspection. If we refer to A as being above B, then we may state that the fourth fiber of the first or third order connects with

the second order fiber of the second group below, as $A4 - C$, $B4 - D$, $C4 - E$, $D4 - A$, $E4 - B$, with similar connections between the second and third order, $C - A4'$, $D - B4'$, etc. In a similar manner the third fiber of the first or third order connects with the second order of the group directly below, as $A3 - B$, $B3 - C$, $C3 - D$, etc. And if we take a group of first order fibers, as the C group, we have $C1 - A$, $C2 - B$, $C3 - D$, and $C4 - E$.

Under the above conditions, a fiber of the first order, as $A3$, can then excite its corresponding third order fiber, $A3'$. However, the latter, $A3'$, may be excited by the simultaneous stimulation of any fiber of group A together with any fiber of group B. In order to meet this objection we shall consider the mechanism from two points of view. First we consider all fibers identical and also that the time during which two impulses can summate at a synapse, or the period of facilitation, is very short (Lorento de Nó, 1938). Then we shall assume that there are two types of fibers and also that properties of the fibers are as discussed by Rashevsky (1938).

If the period of facilitation at the synapses between the second and third orders, as $A - A3'$, is small, then the two impulses initiated by the first order fiber, $A3$, arrive together at the third order synapse, $A3'$ (we assume that the second order fibers are very long as compared with any collaterals), and by summation can exceed the threshold of $A3'$ (Lorento de Nó, 1938); whereas, an impulse initiated by $A1$, $A2$, or $A4$ and an impulse initiated independently by a fiber in the first order of group B would not arrive at $A3'$ within the summation period, except under rather special circumstances, and hence the summation would be insufficient to exceed the threshold of $A3'$. In a similar manner $B2'$ would be excited by $A3$. If either $A3$ or $B2$ are excited both $A3'$ and $B2'$ will be excited. If, in the first and in the third order, two fibers constituting a pair (as $A3$, $B2$) are together, a group with such pairs taken as units will show the properties required — a correspondence between units in the first and third orders: a separate study would be required to investigate the special conditions under which independently initiated impulses would summate, and more specific assumptions regarding the mechanism and the actions of the various fibers would be required.

Let us now assume that the period of summation is long compared with the period between impulses even at low intensities of stimulation. Then let the afferent and efferent ends of the second order fibers be of two kinds. Let those which connect with another group (light lines in Figure 1) be capable of carrying a greater excitation (i.e. having a higher limiting value of E; Rashevsky, 1938, p. 218) than those which connect with the same group (heavy lines). In fact, assume

that the simultaneous maximal stimulation of all of group B together with $A4$ will not quite excite $A3'$, whereas the stimulation of $A3$ alone will produce a sufficiently strong excitation in fiber B which is able to summate with a weaker excitation through A at the synapse to exceed the threshold of $A3'$. Under these conditions there will be a point to point correspondence between the first and third order.

One-half of the mechanism of Figure 1 is drawn in Figure 2 at right angles to the former figure and slightly rearranged. The connections are the same as in Fgure 1 as is also the labeling. The solid circles represent the second order fibers whereas the open circles may represent either the first or third order fibers. The figure is then in the plane of the retina or "central retina" (Rashevsky, 1938, p. 266) and only one complete unit is shown. Though the connections seem rather random at first glance, the same cyclic symmetry is evident when it is looked for.

Whether or not the above principle is found to exist anywhere in the central nervous system, it is worth while to note that a very simple symmetric pattern, as in the figures, may be very obscure and would be difficult to observe unless it was looked for, after being indicated by some hypothetical consideration.

Summary. A particular symmetric arrangement of nerve fibers, which have properties customarily attributed to them, is shown to exhibit the property of point to point correspondence between two sets of excitable elements when the number of intermediate fibers is smaller than the number in either of the connected sets.

II. *A mechanism transmitting nerve impulses in a limited intensity range.* Let a stimulus S be applied to two nerve fibers which arise and terminate together (I of Figure 3); one excitatory (solid)
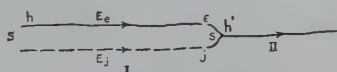

FIGURE 3

and one inhibitory (dotted). Let the thresholds for the two fibers be equal and equal to h. The intensity of excitation in the excitatory fiber, E_e, and in the inhibitory branch, E_j, may be given by (Rashevsky, 1938, Ch. XXII)

$$E_e = \frac{I_e}{\theta_e} \left[1 - e^{-a_e \, \theta_e (S-h)}\right] \tag{1}$$

and

$$E_j = \frac{I_j}{\theta_j} \left[1 - e^{-a_j \, \theta_j (S-h)}\right] \tag{2}$$

where I is the intensity of each individual nerve impulse, θ is the refractory period and α is a constant of proportionality.

At the synapse between the pair of fibers and the next higher order neuron, s, the excitatory and inhibitory fibers produce respectively the excitation factor ε and inhibition factor j according to the equations (Rashevsky, 1938, Ch. XXII)

$$\frac{d\varepsilon}{dt} = A\,E_e - a\,\varepsilon \tag{3}$$

$$\frac{dj}{dt} = B\,E_j - bj\,. \tag{4}$$

Solving equations (3) and (4) and imposing the initial conditions that $\varepsilon = j = 0$ for $t = 0$, we have

$$\varepsilon = \frac{A\,E_e}{a}\,(1 - e^{-at}) \tag{5}$$

$$j = \frac{B\,E_j}{b}\,(1 - e^{-bt})\,. \tag{6}$$

In order for excitation to occur in the higher order nerve fiber II, the net excitation must exceed the threshold, $\varepsilon - j > h'$, or using equations (1), (2), (5) and (6), we have,

$$h' < \frac{A\,I_e}{a\,\theta_e}\,[1 - e^{-a_e\,\theta_e(S-h)}]\,(1 - e^{-at})$$

$$\tag{7}$$

$$-\frac{B\,I_j}{b\,\theta_j}\,[1 - e^{-a_j\,\theta_j(S-h)}]\,(1 - e^{-bt})\,.$$

Of several possibilities, let us here consider the case that

$$\frac{A\,I_e\,a_e}{a} > \frac{B\,I_j\,a_j}{b}\,;\;\frac{A\,I_e}{a\,\theta_e} < \frac{B\,I_j}{b\,\theta_j}\,.$$

These conditions may be written in a somewhat more special form:

$$a_e\,\theta_e >> a_j\,\theta_j\,;\;\frac{A\,I_e}{a\,\theta_e} < \frac{B\,I_j}{b\,\theta_j}\,. \tag{8}$$

Under conditions (8), relation (7) may be written with close approximation for the stationary state

$$h' < \frac{A\,I_e}{a\,\theta_e} - \frac{B\,a_j\,I_j}{b}\,(S - h)\,. \tag{9}$$

Hence, in order that the second order fiber be excited, we must have

the stimulus intensity, S, lie within the range

$$\frac{b}{B\,a_j\,I_j}\left(\frac{A\,I_e}{a\,\theta_e}-h'\right)+h>S>h,\tag{10}$$

which can be satisfied if

$$\frac{A\,I_e}{a\,\theta_e}>h'.\tag{11}$$

Let

$$\delta=\left(\frac{A\,I_e}{a\,\theta_e}-h'\right)\frac{b}{B\,a_j\,I_j}\tag{12}$$

so that equation (10) may be written

$$h+\delta>S>h\tag{13}$$

and only stimuli whose magnitudes lie between h and $h+\delta$ are capable of eliciting excitation in the second order fiber II and that, only if δ is positive. A similar result as this was obtained by postulating a more complex mechanism by N. Rashevsky (1938a).

Now if δ is very small, values of S approximately equal to h are the only ones which produce excitation in II. The width of the range of intensities transmitted, δ, decreases with increasing h'. This is equivalent to stating that the acuity of discrimination increases with increased inhibition (increased threshold) at the synapse between I and II.

We have treated the fiber pair I in Figure 3 as two fibers, one excitatory and the other inhibitory. Evidently they could as well be a single fiber producing two factors, ε and j, which are not only produced at different rates but which also depend in a different way upon the stimulating influence. The results which have been obtained from conditions (8), in particular, would be unchanged, so that such *a general type fiber may, under certain conditions, transmit excitation within a limited range of stimulus intensities.*

Equations (1) and (2) are of a specific form, but evidently more general relations could have been used. If we designate these relations by $E_e(S)$ and $E_j(S)$, then, in order that transmission will occur in a single, finite range we must have [equations (5) and (6) for $t=\infty$]

$$\frac{A}{a}\,E_e(S)-\frac{B}{b}\,E_j(S)>h'\tag{14}$$

in a continuous finite range, $h+\delta_1\geqslant S\geqslant h+\delta_2$, where $h+\delta_1$ and $h+\delta_2$ are the solutions of (14) with the inequality sign replaced by the equality sign.

Summary. If a fiber producing only the excitation factor, ε , excites a fiber of the general type with the properties discussed above, then for the stationary state, only a certain range of frequencies in the first order fiber will be transmitted through the general type fiber to some third order fiber, since ε_1 of the first fiber is determined by the excitation E_1 of the first fiber, which is by assumption (Rashevsky, 1938, Ch. XXII) proportional to the frequency of the impulses. Another first order fiber, having different parameters, would have a different range of frequencies transmitted. And if the two fibers act together, only certain combinations of frequencies could be transmitted.

III.   A mechanism in which transmission moves along a contour. Consider now a large number of identical fiber pairs whose afferent ends are excited by the efferent ends of one or more lower order neurons. Let these points of contact occur in a random manner so that there is a distribution in the number of synapses between the lower order and higher order fibers; and, as a result, the higher order fiber pairs will be excited variously by intensities ranging from zero to some maximum value. Then only those fibers whose stimulus intensities range from h to $h + \delta$ [equation (13)] will transmit the excitation. If now the initial stimulus applied to the lower order neuron is increased, all the intensities exciting the fiber pairs are increased and a different set of fibers will transmit the excitation due to the intensities ranging from h to $h + \delta$.

On the other hand, we may consider a large number of similar fiber pairs, differing only in respect to the threshold. Then, if they are equally excited by some stimulus, only those will transmit excitation whose synaptic thresholds range from $h - \delta$ to h ; or from h_0 to $h_0 + \delta$ if $h_0 \equiv h - \delta$. The net result here is the same as that obtained by the mechanism discussed in the preceding paragraph. More generally, then, we may have a variation in both the number of synaptic contacts between the lower and higher order fibers and the threshold values of the fiber pairs.

If the afferent processes of a large number of fiber pairs (Type I) of various thresholds are in synaptic contact with the innumerable collateral branches of the axon ending of the lower order neuron (cf. neuropil), we have the more general case discussed above. Let the lower order neuron be of the intermediate excitatory type (Rashevsky, 1938, p. 260) and let its excitation and inhibition constants a' and b' be very much smaller than either a or b of the fiber pairs [equations (3) and (4)]. The fiber pairs are then always in a nearly stable position as the net excitation, $(\varepsilon - j)$, in the synapses between the

first order neuron and second order fibers rises and then falls. We may then use equation (13) with S replaced by $\varepsilon - j$ since $\varepsilon - j$ plays the role of a stimulus. But $\varepsilon - j$ first increases from zero rapidly to a maximum, then decreases somewhat more slowly to zero, so that the point of maximum net excitation $(\varepsilon - j)$ is nearer the initial time than the second time at which $\varepsilon - j = h_1$.

If we consider the fiber pairs arranged in the order in which excitation occurs with increasing stimulus strength (for one to one synaptic connections, in order of increasing thresholds h; for equal thresholds, in order of decreasing frequency of synaptic contacts), it follows from the previous discussion that, when the first order neuron is given a constant stimulus, at first the low threshold fiber pairs transmit the excitation for a short time and subsequently higher threshold fiber pairs transmit. The excitation passes over as a wave of variable width from the fibers of lowest thresholds to the fibers whose thresholds are $(\varepsilon - j)_{max}$; the wave of excitation then passes back to the lowest threshold and stops, the return being slower than the forward process. If the secondary fibers are arranged at random in space, the appearance of a wave as such would be entirely obscured. The result would be more analogous to a great many lights, each flashing on momentarily in no apparent order and only slightly overlapping each other in time. The last few to come on would remain on longer; then a complete reversal would take place — the lights flashing in reverse order and at a slower rate, with longer duration of the light periods.

An interesting case arises if the secondary fibers are arranged in order along a contour and if the first order neuron exhibits a slow periodic fluctuation of the excitation factor. (Landahl and Householder, 1939). The transmission of excitation then travels as a wave back and forth along the contour with possible delays at each end. A mechanism with this property was suggested by N. Rashevsky (1938a), in connection with the development of a theory of aesthetic values, as being part of a mechanism for discrimination of detail.

Let a light line segment on dark background be presented and let L be its geometrically similar image on the retina along which the photosensitive elements are excited. Let L_1 be a corresponding image on the central retina (Rashevsky, 1938, p. 266). Let the excited elements along L_1 each send efferent processes in a random manner to a higher center (as discussed above) which is the group of synapses between the latter efferent ends and the afferent ends of fibers of the type discussed in section II; and let these latter fibers be arranged in the order in which excitation occurs with increasing stimulus strength. If the afferent ends of each of these latter fibers (fiber pairs) are also

in synaptic contact with the efferent collaterals of a neural circuit showing periodicity then, under appropriate conditions, transmission will occur through the fiber pairs to a higher order fiber as though an adequate stimulus, of variable width, had been moved back and forth along the line of the afferent ends of the fiber pairs. If for a certain line L_1, there is a specific linear center of the fiber pairs, we then have a possible mechanism exhibiting the properties suggested (Rashevsky, 1938a).

Summary. General fibers, satisfying the conditions discussed in section II, are used in connection with a mechanism which has the property of "following" a contour.

IV. The reaction to the size of an object independent of its distance. It is a well known phenomenon that the size of an object can be recognized within a considerable distance range (Köhler, 1929), and a great number of comparisons influence the judgment. However, let us assume that all cues are eliminated except those obtained from the image of the object on the retina and from the muscles of the eyes, the latter being in focus upon the object. N. Rashevsky (1938, p. 276) has discussed the possibility of interpreting the phenomenon under such conditions. Let us consider the neural mechanism of Figure 4 which has the property of giving a representation of the size of an object irrespective of its distance.

Let proprioceptive nerve fibers which arise at the fibers of a muscle of the eye be collected together into a single tract and let these fibers or their collaterals indirectly influence a group of neurons N. The nerve fibers will have some distribution of thresholds so that for various degrees of contraction or extension of the muscle, various numbers of fibers will be excited and the intensity of excitation, E, in each will vary with the degree of contraction. We shall hereafter refer to each tract of nerve fibers as though a single fiber and treat the group N as a single neuron.

Let the proprioceptive nerve fibers I, II, III, IV arise respectively from the lateral rectus, medial rectus, zonula ciliaris, and ciliaris muscles of the left eye and corresponding fibers (primed) to the right eye (Figure 4). Assume that the corresponding fibers from each eye, as I and I', are equivalent. Let fibers I and III be of the simple inhibitory type producing only j, and let fibers II and IV be of the simple excitatory type producing only ε. We shall also consider only the stationary state so that ε or j is directly proportional to the total excitation, E, in the fiber (Rashevsky, 1938, Ch. XXII).

If the subscripts *1, 2, 3, 4* refer to the fibers I, II, III, IV or the
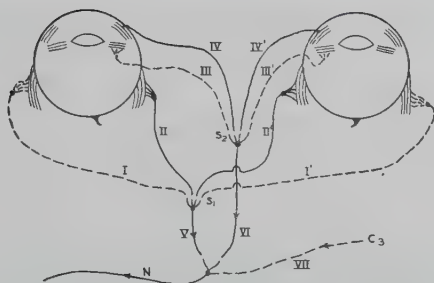
FIGURE 4

corresponding muscles, and if x_i is the degree of contraction of muscle i from some position, then $E_i(x_i)$ is the intensity of excitation in the i-th fiber when the corresponding muscle is contracted by an amount x_i. Considering only those nerve endings which are excited by the contraction of the muscle, we have $E_i(x_{im}) = 0$, where x_{im} is the minimum degree of contraction producing excitation and is a threshold value, and also $E_i(x_i)$ is some function which increases monotonically and is concave downwards. Equations (1) and (2) $[E(S)]$ satisfy these conditions. Let us however assume that, over a sufficiently wide range, a linear relation holds in the case of fibers I and II and that the fibers are equivalent except that one produces j and the other ε, or $x_{1m} = x_{2m} = x_m$ and $\beta_1 = \beta_2 = \beta$, and hence

$$E_1(x_1) = \beta(x_1 - x_m), \tag{15}$$

$$E_2(x_2) = \beta(x_2 - x_m). \tag{16}$$

Let the eyes be focused upon an object which is "straight ahead" and let the distance from the eyes to the object of area s be d. Also let p be the distance from the fovea to the optical center of the lens system, $2l$ be the distance between the eyes, and 2θ the angle, at the object, formed by the lines from the eyes. If σ is the retinal size of the object, we have the relations

$$\frac{\sigma}{s} = \frac{p^2}{d^2}, \tag{17}$$

and

$$\tan\theta = \frac{l}{d} \doteq \sin\theta \doteq \theta, \tag{18}$$

the latter approximations being accurate except for objects within a few inches of the eyes.

Now for an object indefinitely far away let the states of contraction be $\bar{x}_1 = \bar{x}_2 = \bar{x}$ for the muscles corresponding to I and II and let

Δx_i be the change due to convergence of the eyes. Then we have

$$x_1 = \bar{x} + \Delta x_1 \tag{19}$$

$$x_2 = \bar{x} + \Delta x_2 \tag{20}$$

and

$$\Delta x_1 = - \Delta x_2 \tag{21}$$

since

$$x_1 + x_2 = \bar{x}_1 + \bar{x}_2 = \text{constant}, \tag{22}$$

or the degree of extension of one equals the degree of contraction of the other. But, since the change in degree of contraction is proportional to the angle and the latter is given by equation (18),

$$\Delta x_2 = - \Delta x_1 = k\,\theta = \frac{kl}{d}, \tag{23}$$

where k is a fixed parameter depending on the lengths of the muscles and the diameter of the eye. Substituting (23) into (19) and (20) and the results into (15) and (16), we have for each eye

$$E_1(x_1) = \beta\,(\bar{x} - x_m - \frac{kl}{d}), \tag{24}$$

$$E_2(x_2) = \beta\,(\bar{x} - x_m + \frac{kl}{d}). \tag{25}$$

Now $\varepsilon_2 = A_2\,E_2/a_2$ and $j_1 = B_1\,E_1/b_1$; but as the fibers were assumed equivalent $A_2/a_2 = B_1/b_1$. Then at synapse s_1 we have for the next excitation $(\varepsilon - j)$ from the fibers I, II, I' and II', of both eyes,

$$(\varepsilon - j)_1 = 2\,\varepsilon_2 - 2\,j_1 = \frac{4\,A_2\,\beta\,kl}{a_2\,d} = \frac{h_5\,d_{01}}{d} \tag{26}$$

where h_5 is the threshold of V , and $d_{01} = 4\,A_2\,\beta\,kl/a_2 h_5$. Then let the fiber V be such that the intensity of excitation E_5 may be approximated by a logarithmic function of the stimulus (Rashevsky, 1938, p. 264) so that we may write, if K_5 is a constant depending on fiber V ,

$$E_5 = K_5 \log \frac{(\varepsilon - j)_1}{h_5} = K_5 \log \frac{d_{01}}{d}. \tag{27}$$

As the eyes change accommodation from a distant to a closer object, the ciliaris muscle of each eye contracts, increasing the excitation E_4 in the excitatory fibers IV and IV' , and the zonula ciliaris of each eye is relaxed, decreasing the excitation E_5 in the inhibitory fibers III

and *III'* . The actual mechanism here is complex but the net excitation $(\varepsilon - j)_2$ at synapse s_2 increases with the reciprocal of the distance. Let us assume that the result is such that we may approximate the intensity of excitation E_6 in *VI* by an equation similar to (27) or

$$E_6 = K_6 \log \frac{d_{02}}{d} . \tag{28}$$

Let the contour of the retinal image of the object be represented at a center C_3 (Figure 4) as discussed by Rashevsky (1938, Ch. XXVII) so that the net excitation factor is proportional to the length of the contour. For similar retinal images the length is proportional to the square root of the retinal area, σ . A particular value of the excitation at C_3 then represents a particular retinal size, the shape being constant. Now let an inhibitory fiber *VII*, from C_3 to N , also be of a type for which we may use the logarithmic relation between the excitation E_7 in the fiber and the net excitation factor, proportional to $\sqrt{\sigma}$, acting as the stimulus, so that if K_7 and σ_0 are constants,

$$E_7 = 2 K_7 \log \frac{\sqrt{\sigma}}{\sqrt{\sigma_0}} = K_7 \log \frac{\sigma}{\sigma_0} . \tag{29}$$

Since $\varepsilon = A E/a$ and $j = B E/b$ for the stationary state, we have from equations (27), (28), and (29), for the net excitation $(\varepsilon - j)_n = \varepsilon_5 + \varepsilon_6 - j_7$ acting on N ,

$$(\varepsilon - j)_n = \frac{A_5 K_5}{a_5} \log \frac{d_{01}}{d} + \frac{A_6 K_6}{a_6} \log \frac{d_{02}}{d} - \frac{B_7 K_7}{b_7} \log \frac{\sigma}{\sigma_0} . \tag{30}$$

If $A_5 K_5/a_5 = A_6 K_6/a_6 = B_7 K_7/b_7$, we then have for the above equation, because of (17)

$$(\varepsilon - j)_n = \frac{A_5 K_5}{a_5} \log \frac{d_{01} d_{02} \sigma_0}{d^2 \sigma} = \frac{A_5 K_5}{a_5} \log \frac{d_{01} d_{02} \sigma_0}{p^2 s} \tag{31}$$

or

$$(\varepsilon - j)_n = \frac{A_5 K_5}{a_5} \log \frac{s_0}{s} , \tag{32}$$

where $s_0 = d_{01} d_{02} \sigma_0/p^2$. Then under the above conditions, N receives stimulation which depend only on the variable, s , which is the size of the external object.

It will also be noted that if the object is not directly in front of the subject but to one side so that both eyes are turned from being directed straight ahead, it can be seen that the net excitation from *I* and

II will still depend only upon the angle between the eyes if equations (24) and (25) are assumed.

In order to arrive at equation (32) it was necessary to make a number of restrictive assumptions. Let us show that under much more general conditions we may obtain approximately the same result. From the previous discussion it is clear that, on all counts, the values of E_5 and E_6 increase with the reciprocal of the distance d from the object to the eye. Let the total excitation factor acting upon N due to E_5 and E_6 be given by $\varepsilon_a(d)$ which then increases with $1/d$. Similarly the inhibitory factor j_7 due to the center C_3 increases in some manner with the linear retinal size but the latter, for a given object, varies directly with the linear size of the object and inversely with the distance d, and hence $j_7(d)$ is an increasing function of $\sqrt{s}/d$. The net excitation acting on N is then

$$(\varepsilon - j)_n = \varepsilon_a - j_7 \, . \tag{33}$$

Let ε_a and j_7 be given by the series

$$\varepsilon_a = a_0 + a_1 \frac{d_1}{d} + a_2 \frac{d^2_1}{d^2} + \cdots , \tag{34}$$

$$j_7 = b_0 + b_1 \sqrt{\frac{s}{s_0}} \frac{d_2}{d} + b_2 \frac{s}{s_0} \frac{d^2_2}{d^2} + \cdots , \tag{35}$$

where d_1, d_2 and s_0 correspond to threshold values. Also let ξ and η be defined by

$$\xi = \bar{s} - s \, , \tag{36}$$

$$\eta = \bar{d} - d \, , \tag{37}$$

where $\bar{s}$ and $\bar{d}$ are values of s and d which are well removed from extreme values. Then substituting for s and d in terms of ξ and η in (34) and (35), we have upon expanding,

$$\varepsilon_a - j_7 = \alpha + \beta\,\xi + \gamma\,\eta + \cdots , \tag{38}$$

where

$$\alpha = a_0 - b_0 - b_1 \sqrt{\frac{\bar{s}}{s_0}} \frac{d_2}{\bar{d}} - b_2 \frac{\bar{s}}{s_0} \frac{d^2_2}{\bar{d}^2} + a_1 \frac{d_1}{\bar{d}} + a_2 \frac{d^2_1}{\bar{d}^2} \, ,$$

$$\beta = \frac{1}{\bar{s}} \left[\tfrac{1}{2} b_1 \sqrt{\frac{\bar{s}}{s_0}} \frac{d_2}{\bar{d}} + b_2 \frac{\bar{s}}{s_0} \frac{d^2_2}{\bar{d}^2} \right] \, ,$$

$$\gamma = \frac{1}{\bar{d}} \left[a_1 \frac{d_1}{\bar{d}} + 2\, a_2 \frac{d^2_1}{\bar{d}^2} - 2\, \beta\, \bar{s} \right] \, . \tag{39}$$

In equation (38) only the first order terms in ξ and η from the expansions are written, and only those from the first three terms of (34) and (35). If we impose the restriction that

$$\gamma = 0 , \tag{40}$$

we may then write for the net excitation acting on N, from (33), (36) and (38),

$$(\varepsilon - j)_n = \alpha + \beta(\bar{s} - s) + \cdots , \tag{41}$$

so that, to a first approximation, the net excitation factor acting on N is a function of the object's size, s, only. If an object of size $\bar{s}$ at a distance $\bar{d}$ is used as a standard, a similar object of size s and distance d, values not too removed from $\bar{s}$ and $\bar{d}$, will influence N depending only on the relative sizes. The deviation from a judgment of size only will increase as either s or d deviate from the size or distance of the standard and will increase as the standard deviates from $\bar{s}$ and $\bar{d}$. The size and distance of an object at which judgments are determined most completely by size alone are given by $\bar{s}$ and $\bar{d}$.

Summary. A neural mechanism has been discussed which has the property, under certain conditions, of giving a representation of the external size of an object independent of its distance.

V. An interpretation of the effect of a warning stimulus and of stimulus strength upon reaction time. Let us set up a simple abstraction of the situation in which a stimulus produces a fixed response but also in which another stimulus produces a "set". One of the simplest possible abstractions is given in Figure 5 in which the stimulus S pro-
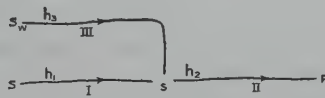

FIGURE 5

duces a response, R, and a warning stimulus S_w influences the response.

Let the subscripts *1*, *2*, and *3* refer to the fibers *I*, *II* and *III* respectively. Also let $E_1(S)$ and $E_3(S_w)$ be the respective monotonically increasing functions of the stimuli, $S > h_1$ and $S_w > h_3$. Since we are considering only values of S sufficient to produce the response without S_w, the net excitation $(\varepsilon - j)$ due to I must increase monotonically from zero to a value h_2 at least. Hence without effecting any essential change we may assume that I is a simple excitatory

fiber producing only ε. In the case of fiber II we need not specify anything except that when the net excitation at synapse s exceeds h_2 a response is produced after a time t_0, a value which we shall assume to be essentially a constant and which includes the conduction time and the delay at the muscle. Fiber III we shall assume to be of the general type producing both ε and j. We then have, similar to equations (3) and (4), the following equations for the excitatory and inhibitory factors at synapse s (Figure 5),

$$\frac{d \varepsilon_1}{d t_1} = A_1 E_1 - a_1 \varepsilon_1 , \tag{42}$$

$$\frac{d \varepsilon_3}{d t_3} = A_3 E_3 - a_3 \varepsilon_3 , \tag{43}$$

$$\frac{d j_3}{d t_3} = B_3 E_3 - b_3 j_3 , \tag{44}$$

where the nerves have different parameters, and where the times t_1 and t_3 are not measured from the same origin. Since the E's are constant we integrate and obtain the relations

$$\varepsilon_1 = \frac{A_1 E_1}{a_1} (1 - e^{-a_1 t_1}), \tag{45}$$

$$\varepsilon_3 = \frac{A_3 E_3}{a_3} (1 - e^{-a_3 t_3}), \tag{46}$$

$$j_3 = \frac{B_3 E_3}{b_3} (1 - e^{-b_3 t_3}), \tag{47}$$

where the constants of integration are determined by the initial conditions $\varepsilon_1 = 0$ for $t_1 = 0$ and $\varepsilon_3 = j_3 = 0$ for $t_3 = 0$, so that the times t_1 and t_3 are measured from the times of presentation of S and S_w respectively, but do not include t_0.

The net stimulus acting on II is then

$$\varepsilon_1 + \varepsilon_3 - j_3 = \frac{A_1 E_1}{a_1} (1 - e^{-a_1 t_1}) +$$

$$\frac{A_3 E_3}{a_3} (1 - e^{-a_3 t_3}) - \frac{B_3 E_3}{b_3} (1 - e^{-b_3 t_3}). \tag{48}$$

The time t_r between the presentation of the stimulus S and the initiation of the response, $t'_1 + t_0$, is given by the solution of equation (48) for $t_1 = t'_1$ when $\varepsilon_1 + \varepsilon_3 - j_3 = h_2$, and adding t_0. Solving for t'_1, we have

$$t'_1 = -\frac{1}{a_1} \log[1 - \frac{a_1}{A_1 E_1} \{h_2 - \frac{A_3 E_3}{a_3} (1 - e^{-a_3 i'_3}) +$$

(49)

$$\frac{B_3 E_3}{b_3} (1 - e^{-b_3 t'_3}) \}].$$

In the above equation t'_3 is the time between the presentation of S_w and the beginning of the reaction, less the time t_0. If the preparatory interval, $t_w = t'_3 + t'_1 + t_0$, is much greater than the reaction time, as is generally the case, we may neglect $t'_1 + t_0$ and refer to t'_3 as the preparatory interval. However, for very short warning periods, the proper correction must be made.

In order that S produce the response in a finite time and that S_w alone should not do so, t'_1 must be positive and finite for all values of t'_3. We then have the condition

$$h_2 > \frac{A_3 E_3}{a_3} (1 - e^{-a_3 t'_3}) - \frac{B_3 E_3}{b_3} (1 - e^{-b_3 t'_3}) >$$

(50)

$$h_2 - \frac{A_1 E_1}{a_1}$$

Equation (49) gives the response time, $t_r = t'_1 + t_0$ as a function of the warning period, t'_3, the strength of the stimulus S in terms of $E_1(S)$, and the strength of the warning stimulus S_w through $E_3(S_w)$, in terms of the parameters a_1, A_1, a_3, A_3, b_3, B_3, h_1, h_2, h_3, and t_0. From the equation we find that the response time t_r decreases with increasing stimulus intensity S such that for a minimum value of S there is no response and for large values of S the time t_r decreases to a minimum value, since $E_1(\infty)$ is finite. Also t_r decreases with the intensity of S_w from a finite initial value to a minimum value, if t'_3 is not too large, in which case t_r may increase with S_w from its initial value and approach a limit. The effect of t'_3 upon the time t_r is best illustrated by the curves below. It should be noted here, however, that the value of t_r for $t'_3 = 0$ and that for $t'_3 = \infty$ are not necessarily equal although they are so taken in the graphs.

Data by H. Woodrow (1914) give the relation between t_r and t'_3 and the relation between t_r and S. A rather complete check could be obtained but unfortunately the values of S were not measured. Hence separate sets of curves will be used to illustrate the above relations.

In order to determine $t_r(S)$ it is necessary to specify $E_1(S)$. That the relationship is approximately logarithmic over a wide range for single fibers is indicated by data of H. K. Hartline and C. H. Graham (1932). Such a relation has also been used in previous theoretical

developments (Rashevsky, 1938), (Landahl, 1938). If we then introduce

$$E_1 = K \log_{10} S - K E_0 ,\qquad(51)$$

where $K E_0$ takes care of the unit of intensity which is generally not given, we have for equation (49)

$$t_r = t_0 - \frac{1}{a_1} \log \left[1 - \frac{H}{\log_{10} S - E_0}\right]$$

or

$$t_r = t_0 + \frac{1}{a_1} \log_e \left[\frac{\log_{10} S - E_0}{\log_{10} S - E_0 - H}\right] ,\qquad(52)$$

where

$$H = \frac{a_1}{A_1 K} \left\{ h_2 - \frac{A_3 E_3}{a_3} (1 - e^{-a_3 t'_3}) + \right.$$

$$\left. \frac{B_3 E_3}{b_3} (1 - e^{-b_3 t'_3}) \right\} .\qquad(53)$$

Equation (52), which is similar to the first half of Rashevsky's, equation (12) (Rashevsky, 1938, Ch. XXII), is compared with experiment as shown in Figures 6 and 7. The data in Figure 6 are by G. O. Berger (1886) and J. McK. Cattell (1886) for subjects B and C. The

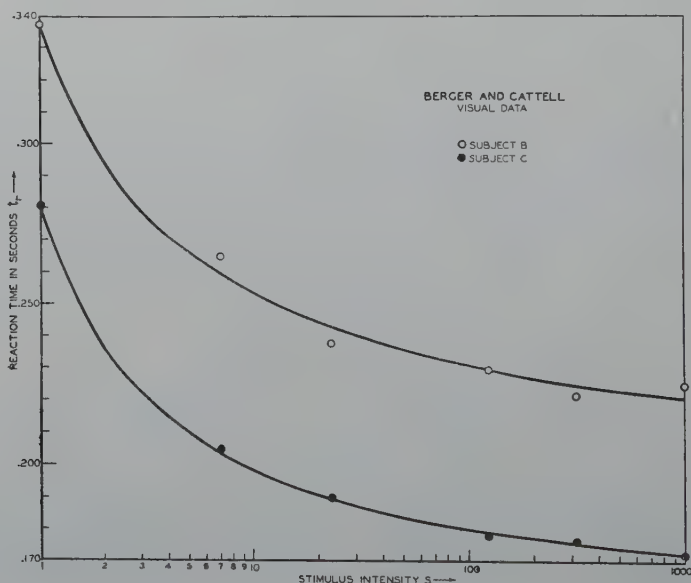BERGER AND CATTELL
VISUAL DATA

O SUBJECT B
● SUBJECT C

FIGURE 6

stimulus intensities were measured photometrically. The range of intensity is from 1 to 1000 so that a logarithmic scale is used. The lowest intensity was near the threshold, being occasionally unnoticed. Each experimental point was the average of approximately one hundred and fifty reactions. A warning signal of a constant time interval was used. The units of intensity are not given, the smallest being given the value 1. For the upper curve the following values were assigned to the parameters in equation (52) to obtain the curve: $t_0 = .193$ sec., $1/a_1 = .413$ sec., $E_0 = -.85$, $H = .25$ (subject B) ; and for the lower curve: $t_0 = .153$ sec., $1/a_1 = .279$ sec., $E_0 = -.68$, $H = .25$ (subject C).

The threshold, S', is given by $\log S' = E_0 + H$. For subject B, the threshold assumed is $S' = \frac{1}{4}$, and for subject C, the threshold is $S' = \frac{3}{8}$. The actual thresholds are not given but are not much less than 1. The above values may possibly be too low.

A large number of experiments under various conditions have been performed by H. Piéron (1920), most of which were visual reactions. For most of the cases about twenty reactions were averaged for each value. Although the number of reactions is smaller for each value, the results are very significant in view of the wider range of intensities, in several cases, and the variety of situations investigated.

Visual data by S. Froeberg (1907) when plotted on a logarithmic scale show a linear relationship. The range is from 1 to 100 and each value is an average of 400 reactions. The threshold was considerably
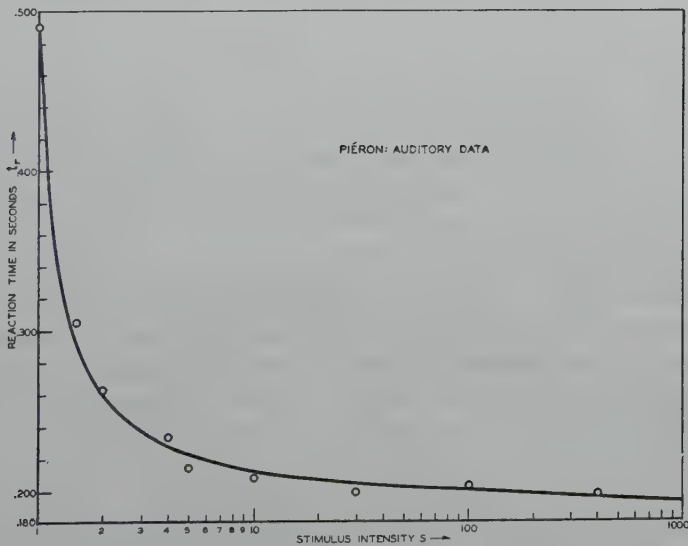

FIGURE 7

less than 1, the smallest value used. The linear relationship between the reaction time and the logarithm of the intensity is contradicted by the results of G. O. Berger (1886) and McK. Cattell (1886) and also by the numerous experiments of H. Piéron (1920), in one case covering a range from 1 to 100,000. It should be noted that the curve must be concave upward for small stimulus intensities if a threshold exists or unless there is a discontinuity. Also, because of the method used to measure the intensity, a slight difference in background from absolute black would shift the measurements so as to make the resulting curve more like that in Figures 6 and 7. The large intensities would show no shift while near the smallest values used the shift would be of the order of 100 times the background intensity.

In Figure 7 auditory reaction times are plotted against intensity on a logarithmic scale (Piéron, 1920). The curve is computed from equation (52) using the following values for the parameters: $t_0 = .185$ sec., $1/a_1 = .304$ sec., $E_0 = -.16$, $H = .10$. The threshold is taken as $S' = .87$, whereas unity is given as the threshold.

In Figure 8 gustatory reaction times are plotted against inten-
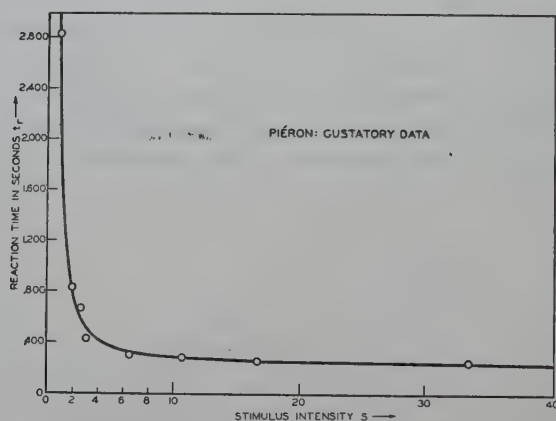

PIÉRON: GUSTATORY DATA

FIGURE 8

sity (Piéron, 1920). The scale in this case is linear. Also, in order to fit the curve so as to include the first point, $S = 1$, it was necessary to assume a linear relation between E and S, that is, $E = \beta(S - h_1)$ (Rashevsky, 1938, Ch. XXII, eq. 3), instead of equation (51) so that instead of equation (52) we have from (49), if $H_0 = H K/\beta$,

$$t_r = t_0 + \frac{1}{a_1} \log_e \frac{S - h_1}{S - h_1 - H_0}. \tag{54}$$

The curve in Figure 8 is obtained by the above equation where the

parameters have the values: $t_0 = .220$ sec., $1/a_1 = 3.800$ secs., $h_1 = .6$, $H_0 = .2$. The value of $S = 1$ is given as the threshold (Piéron, 1920) whereas we have assumed a value of $S' = .80$.

The range for the gustatory data, Figure 8, is from 1 to 33. Unfortunately the range could not be extended. The use of the linear relation between E and S then depends entirely upon the first point. Were it possible to extend the results for higher intensities and to insure the correctness of reaction times for low values of the intensity it would be possible to determine whether a logarithmic or linear relation between E and S would be most accurate over a wide range. It might possibly be that all the relations are very nearly logarithmic or if not, only certain types are linear, as perhaps cases of direct chemical stimulation.

Since the values of the reaction time for low intensities are so important the following suggestion may be considered. If, instead of the mean of a group of reaction times, we use the median, certain advantages are obtained. For large stimuli and short times the mean and median will be very nearly equal. However, for small values of the intensity the situation will be different. For values even several times larger than the threshold no response may be made, or if made, may be excessively long. In either case the mean is not meaningful. If the median is used, it becomes possible to obtain time values almost up to the threshold value, the latter being then taken as a value for which the response is made in infinite time for just one half the trials. For any greater stimulus value the median is finite. Consider a response as having been made if half the trials or more are responded to or noticed and let the time be the median of the response times. Then the threshold is that value for which response is made in infinite time, and it corresponds to the definition in equation $S' = E_0 + H$. On the other hand the mean is infinite unless all failures to react are arbitrarily eliminated. And one excessively high value entirely outweighs the remaining values and it becomes difficult to define an experimental quantity corresponding to the threshold in the equation.

For a constant stimulus the $t_r(t'_3)$ relation may be written from (49)

$$t_r = t_0 - \frac{1}{a_1} \log [M + J(e^{-b_3 t'_3} - e^{-a_3 t'_3})] \qquad (55)$$

where

$$M = 1 - \frac{a_1 h_2}{A_1 E_1} \qquad (56)$$

$$J = \frac{A_3 E_3 a_1}{A_1 E_1 a_3} = \frac{B_3 E_3 a_1}{A_1 E_1 b_3} \qquad (57)$$

It will be noted that in (57) we have assumed that t_r is the same for $t'_3 = 0$ and $t'_3 = \infty$, a restriction which is not necessary but which has some justification.

In the experiments by H. Woodrow (1914) bearing on the above relation three subjects were used. The data from one subject were chosen for illustration in Figure 9. The particular subject, V_s, was
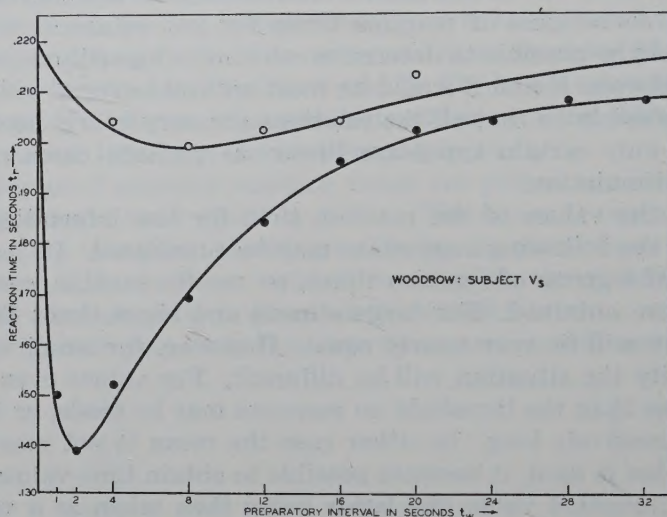

WOODROW: SUBJECT V_s

FIGURE 9

chosen because the curve resulting from his performance is most typical of the three (see Woodrow's statement p. 24) and because the data were most complete. That the trend here is significant has been noted by other writers (Woodrow, 1914, p. 16). The two curves in the figure were obtained under conditions differing in one respect. For the case illustrated by the upper curve the subject did not know the length of the warning period whereas in the other case the subject was given practice with a particular warning period after which his responses with that warning period were recorded.

The mechanism shown in Figure 5 has no hysteresial element and hence as such could not account for any learning. In fact, the warning stimulus has to be assumed to be maintained throughout the interval. The addition of a simple conditioning circuit of either the single or double type (Rashevsky, 1938), (Landahl and Householder, 1938) would eliminate the restriction without necessarily complicating the results appreciably. However, the conditioning of the warning stimulus by the response is evidently of great importance. A more complete treatment of the problem would require a mechanism which, without

conditioning, would give the upper curve in the figure, and which, when conditioning is fairly complete, would give the lower curve.

Pending such a solution let us complicate the mechanism just enough to justify the comparison of the theoretical properties of the mechanism with experiment under analogous conditions. Let an efferent collateral from III (Figure 5) lead to a conditioning center C. Let an efferent collateral from II also lead to C and from C let a fiber, IV, of the type of III lead to the synapse s so that S_w tends to be conditioned to the response. However, let the parameters be such that the net excitation at s due to both III and IV is insufficient to produce the response. Now if the warning periods, t_w, are of random length, no conditioning takes place and fiber IV carries no excitation. Equation (55) then may be used. If, however, a given warning period tends to be conditioned to the response, both III and IV influence synapse s. The additional excitation due to IV decreases the time reaction t_r. If the effect of IV is much greater than III, the form of equation (55) may again be used but quite different parameters will hold.

For the upper curve in Figure 9, the following values of the parameters in equation (55) were used: $t_0 = .130$ sec., $1/a_1 = .200$ sec., $M = .638$, $J = 2.00$, $a_3 = .132$ sec^{-1}., $b_3 = .120$ sec^{-1}. For the lower curve in Figure 8 the values of the parameters in equation (55) were: $t_0 = .130$ sec., $1/a_1 = .200$ sec., $M = .670$, $J = .41$, $a_3 = 1.28$ sec^{-1}., $b_3 = .128$ sec^{-1}. The data were obtained by Woodrow (1914) from subject V_s. Each of the upper set of points represents an average of 100 reactions and each of the lower set represents an average of 150 reactions. Note that t_0 and a_1 are taken equal for each case and that M and b_3 are nearly equal in each case. The value of $1/a_1$ is of the same order of magnitude as that in the cases of visual and auditory stimuli, Figures 6 and 7. The upper curve could not be made to pass nearer the highest and last point. A change in a_1, which would not be easily justified, might raise the curve at the right. However, the average curve for the three subjects does not show such a sudden rise between $t_w = 16$ and $t_w = .20$.

It is of interest to note that the value of $1/a_1$ is of the same order of magnitude for the above curves (Figure 9) as that for the visual data (Figure 6) and auditory data (Figure 7), although there is no apparent connection between the curves. However, it should be noted here that the intensity time curves are largely determined by the threshold S', the value t_0 and the product of H and $1/a_1$. As long as H/a_1 remains constant changes in H or $1/a_1$ do not appreciably affect the shape of the curve. Then making the values $1/a_1$ the same for two different situations imposes a condition upon H. If then any one is determined independently, the others are fixed.

Summary. The properties of a simple neural mechanism (Figure 5) are developed theoretically and applied to an experimental situation. The development leads to a relation [equation (49)] among the experimental variables S, the stimulus intensity, t_r, the reaction time, and t_w, the warning period. The relation between the reaction time and stimulus intensity, derived from the general relation, is compared with experimental data in Figures 6, 7 and 8. The relation between the reaction time and the warning period, also contained in the general relation, is compared with experiment in Figure 9.

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